Strategies for Testing Series

It can be difficult to know which test to use when trying to determine if a series converges or diverges. It takes lots of **practice** to be able to recognize when to use each test. Asking yourself some of these questions might help you to better understand which test to use.

Does the series look like a p-series?

If so, use the comparison test with a simple p-series.

Example: $\sum \frac{10}{7n^4 + 3n + 1}$

Compare this series to the p-series $\frac{10}{7}\sum \frac{1}{n^4}$

$\frac{10}{7n^4 + 3n + 1} < \frac{10}{7n^4}$

We know the series $\sum \frac{1}{n^4}$ converges, because it is a p-series with $p > 1$.

Therefore, by the comparison test, our series converges.

Does the series look like a geometric series?

Example: $\sum 2^{3n} 4^{1-n}$

We can rearrange this equation into the form of a geometric series.

$2^{3n}4^{1-n} \rightarrow 8^n 4^{-(n-1)} \rightarrow \frac{8^n}{4^{n-1}} \rightarrow 8(2)^{n-1}$

In this geometric series, $a = 8$ and $r = 2$. r > 1, therefore the series diverges.

Can you quickly determine that the $\lim_{n\to\infty} a_n \neq 0$?

If so, the series diverges.

Example: $\sum_{n=1}^{\infty} \frac{n^3}{4n^3 + 3}$

$\lim_{n\to\infty} \frac{n^3}{4n^3 + 3} \rightarrow \lim_{n\to\infty} \frac{1}{4 + \frac{3}{n^3}} = \frac{1}{4}$

$\frac{1}{4} \neq 0$, therefore the series diverges.

Does the series contain $(-1)^n$?

Try the alternating series test.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

$\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, therefore the series converges.

Does the series contain a factorial or a constant raised to the *n*th power?

Try the ratio test.

Example: $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^2}{2^{n+1}}}{\frac{(-1)^n n^2}{2^n}}\right| = \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}\left(1+\frac{1}{n}\right)^2$$

$\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1$, therefore the series converges.

Is the series in the form of a_n^n ?

Try the root test.

Example: $\sum_{n=1}^{\infty}\left(\frac{n+2}{2n+5}\right)^n$

$$\sqrt[n]{|a_n|} = \frac{n+2}{2n+5}$$

$\lim_{n\to\infty}\frac{n+2}{2n+5} = \lim_{n\to\infty}\frac{1+\frac{2}{n}}{2+\frac{5}{n}} = \frac{1}{2} < 1$, therefore the series converges.

Is the function positive and decreasing? Is the integral $\int_n^{\infty} f(x)dx$ easy to compute?

Try the integral test.

Example: $\sum_{n=1}^{\infty}\frac{1}{x^2+1}$

$$\int_1^{\infty}\frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_1^{t}\frac{1}{x^2+1}dx = \lim_{t\to\infty}\ \tan^{-1}x\Big]_1^t = \lim_{t\to\infty}\left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$

The integral converges, therefore the series converges.

For more help and practice problems, check out these links!

- http://www.math.hawaii.edu/~ralph/Classes/242/SeriesConvTests.pdf
- https://www.youtube.com/watch?v=DvadVYHf3pM
- http://tutorial.math.lamar.edu/problems/calcii/SeriesIntro.aspx
- http://archives.math.utk.edu/visual.calculus/6/series.15/

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Stewart, J. (2008). *Calculus: Early Transcendentals*. 6th Ed. Thomson Brooks/Cole.