Finding the area between two curves

$$A = \int_a^b \left[f(x) - g(x) \right] dx \qquad (1a)$$

Equation (1a) is used to find the area ***A*** between the curves ***y* = *f*(*x*)** and ***y* = *g*(x)** and lines ***x* = *a*** and ***x* = *b*** where *f*(*x*) and *g*(x) are **continuous functions** and ***f*(*x*) ≥ *g*(x)** within the bounds [*a*,*b*].

Equation (1a) can also be represented by equation (1b) where $\boldsymbol{y_U}$ is the upper boundary and $\boldsymbol{y_L}$ is the lower boundary.

$$A = \int_a^b \left[y_U - y_L \right] dx \qquad (1b)$$

Example 1: Find the area bound above by $\boldsymbol{y = x^2 + 2}$, below by ***y* = *x***, and on the left and right by ***x* = 0** and ***x* = 2.**

1) Define y_U and y_L

This problem states them for you!

$y_U = x^2 + 2$

$y_L = x$

2) Set up integral

$$A = \int_0^2 \left[\left(x^2 + 2\right) - (x) \right] dx$$

3) Solve

$$A = \int_0^2 \left[x^2 + 2 - x \right] dx$$

$$A = \left. \frac{x^3}{3} + 2x - \frac{x^2}{2} \right]_0^2$$

$$A = \left[\frac{(2)^3}{3} + 2(2) - \frac{(2)^2}{2} \right] - \left[\frac{(0)^3}{3} + 2(0) - \frac{(0)^2}{2} \right]$$

$$A = \frac{8}{3} + 2 - 0$$

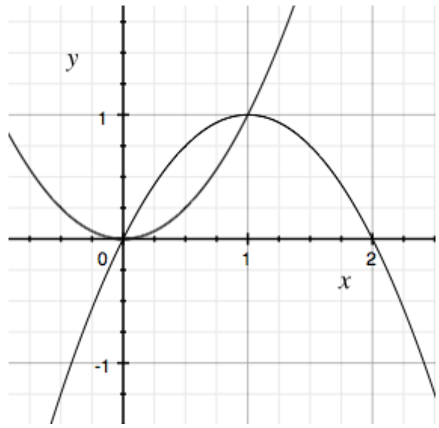
$$A = \frac{14}{3}$$

Example 2: Find the area bound by the curves $y = 2x - x^2$ and $y = x^2$.

1) Graph the functions

This problem doesn't tell you which is your upper and lower bound, so it is important to graph the functions to get a better idea of what these curves look like.

$y = x^2$



$y = 2x - x^2$

2) Define y_U and y_L

Now that you've graphed the functions you can figure out your upper and lower bounds.

$y_U = 2x - x^2$

$y_L = x^2$

3) Define the boundaries of *x*

This problem didn't state any boundaries of *x*, so you have to figure them out yourself. That means you need to figure out where the two curves intersect. Do this by setting the two curves equal to each other and solve for *x*.

$$x^2 = 2x - x^2$$

$$0 = 2x - 2x^2$$

$$0 = 2x(1 - x)$$

$$x = 0,1$$

4) Set up integral

$$A = \int_0^1 \left(\left[2x - x^2\right] - \left[x^2\right] \right) dx$$

$$A = \int_0^1 \left(2x - 2x^2\right) dx$$

$$A = 2\int_0^1 \left(x - x^2\right) dx$$

5) Solve

$$A = 2\int_0^1 \left(x - x^2\right) dx$$

$$A = 2\left(\left. \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \right)$$

$$A = 2\left(\left[\frac{1}{2} - \frac{1}{3}\right] - \left[\frac{0}{2} - \frac{0}{3}\right] \right)$$

$$A = 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

But what if *x* is given as a function of *y*?

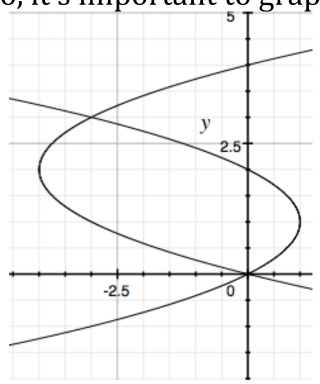
Use Equation 2, where x_R is the right bound and x_L is the left bound.

$$A = \int_c^d (x_R - x_L)\,dy \qquad (2)$$

Example 3: Find the area bound by the curves $\mathbf{x = 2y - y^2}$ and $\mathbf{x = y^2 - 4y}$.

1) Graph the functions

Like Example 2, this problem doesn't tell you which curve corresponds to which bound. So, it's important to graph the functions to determine x_R and x_L.



2) Define x_R and x_L

$x_R = 2y - y^2$

$x_L = y^2 - 4y$

3) Determine the boundaries of *y*

Like Example 2, this problem did not give you the bounds for integration, so you have to find them yourself. To do this, find where the two curves intersect. **Remember, because we are integrating in terms of *y*, the bounds must be values of *y*.**

$$2y - y^2 = y^2 - 4y$$

$$0 = 2y^2 - 6y$$

$$0 = 2y(y - 3)$$

$$y = 0,3$$

4) Set up integral

$$A = \int_0^3 \left[\left(2y - y^2\right) - \left(y^2 - 4y\right)\right]dy$$

$$A = \int_0^3 \left(6y - 2y^2\right)dy$$

$$A = 2\int_0^3 \left(3y - y^2\right)dy$$

5) Solve

$$A = 2\int_0^3 \left(3y - y^2\right)dy$$

$$A = 2\left(\frac{3y^2}{2} - \frac{y^3}{3}\Big]_0^3\right)$$

$$A = 2\left(\left[\frac{3(3)^2}{2} - \frac{3^3}{3}\right] - \left[\frac{3(0)^2}{2} - \frac{0^3}{3}\right]\right)$$

$$A = 2\left(\frac{27}{2} - \frac{27}{3}\right) = \frac{27}{3} = 9$$

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Stewart, J. (2008). *Calculus: Early Transcendentals*. 6th Ed. Thomson Brooks/Cole.